

Brownian Motion & Itô Calculus

Session 19 · The foundation everyone needs — intuition for both tracks

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Primary Text: Liquidity Illusion (Forthcoming, 2026)

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What we'll cover today

1

Random walks → Brownian motion

From discrete to continuous

2

Properties of Brownian motion

Continuous but nowhere differentiable

3

Why ordinary calculus fails

The $dW^2 = dt$ trick

4

Itô's lemma

Stochastic chain rule

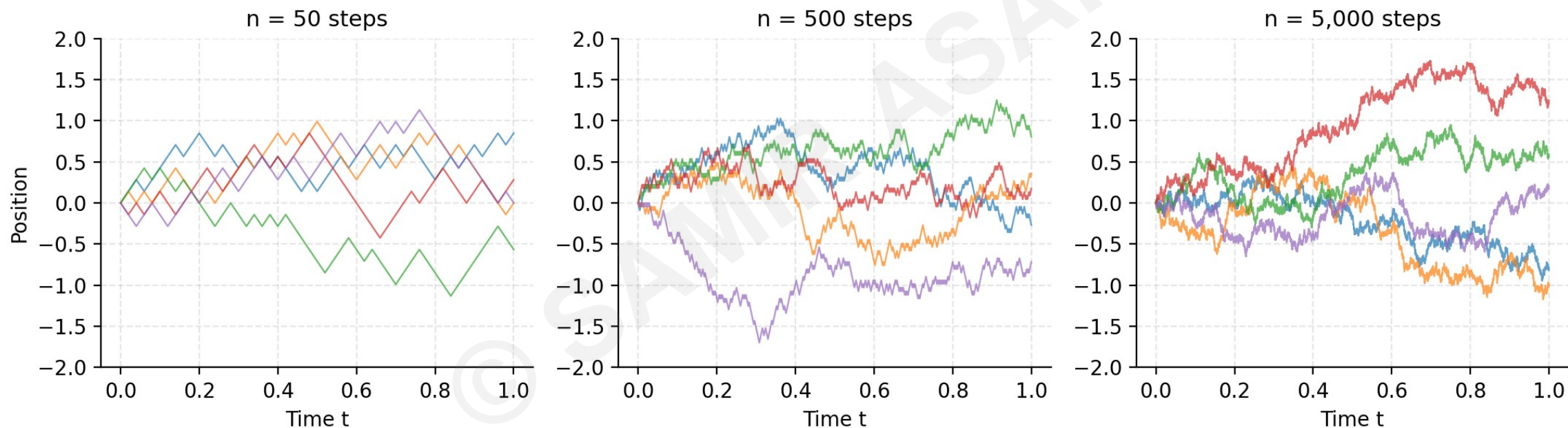
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Application: discount factor dynamics

Where we'll use this in GE-LAV

From random walks to Brownian motion

Random walks → Brownian motion as step size shrinks



Brownian motion $W(t)$ is the limit

Three defining properties:

- $W(0) = 0$
- $W(t) - W(s) \sim N(0, t-s)$
- Independent increments

Why ordinary calculus fails on Brownian motion

Brownian motion is continuous everywhere but differentiable nowhere. Standard calculus doesn't apply.

The fundamental trick: $(dW)^2 = dt$, not 0

In ordinary calculus, $(dt)^2 \rightarrow 0$ faster than $dt \rightarrow 0$, so we ignore it. In stochastic calculus, the variance of dW grows like \sqrt{dt} , so $(dW)^2 \sim dt$ — same order!

STANDARD CHAIN RULE

$$df = f'(x) dx$$

Linear in dx (one term)

ITÔ'S LEMMA

$$df = f'(x) dx + \frac{1}{2} f''(x) (dx)^2$$

Extra second-order term (the Itô correction)

Application: where Itô shows up in GE-LAV

USE 1

OU process derivation

The OU SDE $dL = \kappa(\bar{L} - L)dt + \sigma dW$ is solved using Itô's lemma applied to the integrating factor $\exp(\kappa t) \cdot L(t)$.

→ Session 25 (Track 2)

USE 2

Discount factor dynamics

Applying Itô to $f(r) = \exp(-rT)$ reveals the Jensen convexity correction term — the source of Failure 2.

→ Session 24 (Jensen)

USE 3

HJB equation derivation

The Hamilton-Jacobi-Bellman equation for optimal exit timing comes from applying Itô to the value function $V(L,t)$.

→ Session 20 (HJB)

Track 2 students: read Karatzas-Shreve Ch.3-4. Track 1 students: intuition above is sufficient.

Why we need stochastic calculus

What ordinary calculus can't handle.

Standard calculus

$f(x)$ deterministic; differentiable; well-behaved

Issue

Asset prices follow random walks; not classically differentiable

Need

A calculus where $dW = \sqrt{dt}$ makes sense

Itô's contribution (1944)

Defined the stochastic integral; chain rule for randomness

Why GE-LAV needs this

L_t is an SDE; computing dV requires Itô

Today's goal

Intuition for Itô; not full rigor (that's PhD)

Session 19 summary

What we accomplished today

- 1 Brownian motion is the continuous limit of random walks — fundamental object of stochastic calculus
- 2 BM is continuous but nowhere differentiable. Standard calculus doesn't apply.
- 3 Itô's lemma is the stochastic chain rule — adds a second-order term ($\frac{1}{2}f'' \cdot (dx)^2$)
- 4 GE-LAV uses Itô in three places: OU derivation, Jensen correction, HJB equation

Next session

Session 20: HJB equation and optimal stopping — the math behind $L^*(t)$

Brownian motion: properties

What W_t is.

Definition

Continuous-time process with $W_0 = 0$

Stationary increments

$W_t - W_s \sim N(0, t-s)$

Independent increments

$W_t - W_s$ independent of $W_u - W_v$ for $u < v < s < t$

Continuous paths

Almost surely continuous

Nowhere differentiable

Paths are continuous but jagged at every point

Quadratic variation

$(dW)^2 = dt$ — the key fact for Itô

Itô's lemma: the chain rule for SDEs

How a function of an SDE evolves.

If $dX = \mu \cdot dt + \sigma \cdot dW$

Standard SDE

Then for $f(X,t)$

$$df = (\partial f / \partial t) dt + (\partial f / \partial X) dX + \frac{1}{2} (\partial^2 f / \partial X^2) (dX)^2$$

Key term

$$(dX)^2 = \sigma^2 \cdot dt \text{ — the Itô correction}$$

Substitute

$$df = [\partial f / \partial t + \mu \cdot \partial f / \partial X + \frac{1}{2} \sigma^2 \cdot \partial^2 f / \partial X^2] dt + \sigma \cdot \partial f / \partial X \cdot dW$$

Drift part

$$[\text{brackets}] \cdot dt$$

Volatility part

$$\sigma \cdot \partial f / \partial X \cdot dW$$

Worked example: geometric Brownian motion

The classic. Used in Black-Scholes.

SDE

$$dS = \mu S \cdot dt + \sigma S \cdot dW$$

Take $f(S) = \ln(S)$

$$df = ?$$

Apply Itô

$$df = (\partial f / \partial S) dS + \frac{1}{2} (\partial^2 f / \partial S^2) (dS)^2$$

Compute

$$= (1/S) \cdot (\mu S \cdot dt + \sigma S \cdot dW) + \frac{1}{2} (-1/S^2) \cdot \sigma^2 S^2 \cdot dt$$

Simplify

$$= (\mu - \sigma^2/2) \cdot dt + \sigma \cdot dW$$

Solution

$$\ln(S_t) = \ln(S_0) + (\mu - \sigma^2/2)t + \sigma \cdot W_t \Rightarrow S_t \text{ lognormal}$$

The OU process via Itô

Our L_t process for GE-LAV.

SDE

$$dL = \kappa(\bar{L} - L) \cdot dt + \sigma \cdot dW$$

Solution

$$L_t = \bar{L} + (L_0 - \bar{L})e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} \cdot dW_s$$

Mean

$$E[L_t] = \bar{L} + (L_0 - \bar{L}) \cdot e^{-\kappa t} \rightarrow \bar{L}$$

Variance

$$\text{Var}(L_t) = \frac{\sigma^2}{2\kappa} \cdot (1 - e^{-2\kappa t}) \rightarrow \frac{\sigma^2}{2\kappa}$$

Stationary distribution

$$L_\infty \sim N(\bar{L}, \frac{\sigma^2}{2\kappa})$$

Half-life

$$\ln(2)/\kappa \approx 1.5 \text{ yrs for } \kappa=0.45$$

Itô vs Stratonovich integration

Two conventions; we use Itô.

Itô integral

$\int X \cdot dW$ evaluated at left endpoint of each interval

Stratonovich integral

$\int X \circ dW$ evaluated at midpoint

Numerical difference

$Itô = \text{Stratonovich} - \frac{1}{2} \cdot \int (\partial X / \partial W) \cdot dt$

Why Itô in finance

Aligns with how trading decisions are made (at time t , no peek at $t+dt$)

Stratonovich uses

Physics; smooth-trajectory limits of differential equations

For GE-LAV

Itô throughout • all formulas in this course

Martingale property

Why this matters for asset pricing.

Definition

M_t is a martingale if $E[M_t | \mathcal{F}_s] = M_s$ for $s < t$

Stochastic integral is martingale

$\int X \cdot dW$ is a martingale (under regularity)

In asset pricing

Discounted price processes must be martingales under risk-neutral measure

GE-LAV connection

$V(L,t) \cdot \exp(-rt)$ is a martingale in the continuation region

Optional stopping theorem

Martingale value at optimal stopping time = initial value

Why this matters

Foundation for HJB derivation (Session 20)

Numerical simulation of SDEs

How you'd actually simulate L_t on a computer.

Euler-Maruyama

$$L_t(t+dt) = L_t + \kappa(\bar{L} - L_t) \cdot dt + \sigma \cdot \sqrt{dt} \cdot \epsilon, \epsilon \sim N(0,1)$$

Time step

$dt = 1/252$ (daily) or $1/52$ (weekly) typical for L

Convergence

Weak order 1 for Euler-Maruyama

Higher-order schemes

Milstein (order 1 strong), Runge-Kutta types

Validation

Compare simulated stationary distribution to $N(\bar{L}, \sigma^2/(2\kappa))$

Code

20 lines of Python; in book companion materials

Bridge to Session 20

We can now write SDEs. How do we make decisions?

Decision under uncertainty

Need to optimize over policies, given SDE dynamics

Tool

Hamilton-Jacobi-Bellman (HJB) equation

Two types

Continuous control (allocation) vs discrete (stopping)

Our case

Optimal stopping: when to exit a fund position?

Connection

$L^*(t)$ is the optimal stopping boundary

Session 20 derives

HJB and the smooth-pasting principle