UNIT 5 · MATHEMATICAL INTUITION BRIDGES

HJB Equation & Optimal Stopping

Session 20 · The math behind $L^*(t)$ — when to exit

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What we'll cover today

1. **Why optimal stopping matters**
   Recap from Session 12
2. **The exit decision as a free-boundary problem**
   Continue or stop
3. **HJB equation**
   Bellman's principle for continuous time
4. **Free-boundary conditions**
   Value matching + smooth pasting
5. **Numerical solution preview**
   How $L^*(t)$ is computed in practice

Why optimal stopping is the right framework

Recall Session 12: the trapped investor's problem.

At every moment, an LP chooses between:

CONTINUE (HOLD)

Keep the position. Receive expected future cash flows minus the time-value of capital.

Value = E[continuation payoff | current L]

STOP (EXIT)

Sell now in the secondary market at the current state-dependent price $\pi(L, T\text{-}t)$.

Value = NAV · (1 - $\pi(L, T\text{-}t)$)

The exit boundary $L^*(t)$ is the value of L at which both options are equally valuable.

The Hamilton-Jacobi-Bellman equation

Continuation value V(L,t) satisfies the HJB PDE in the continuation region:

$$\frac{\partial V}{\partial t} + \kappa(\bar{L} - L)\frac{\partial V}{\partial L} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial L^2} - r(L)V + CF(L,t) = 0$$

Each term has economic meaning:

$\partial V/\partial t$	Time decay of optionality	**$\kappa(\bar{L}-L)\cdot\partial V/\partial L$**	Expected change from mean reversion
$½\sigma^2\cdot\partial^2 V/\partial L^2$	Itô correction (uncertainty effect)	**$-r(L)\cdot V$**	Time-cost of capital
$CF(L,t)$	Cash flows received while holding		

Free-boundary conditions: value matching & smooth pasting

At the boundary L*(t) where exit becomes optimal, two conditions must hold:

1. VALUE MATCHING

$$V(L^*(t), t) = NAV \cdot (1 - \pi(L^*(t), T - t))$$

Continuation value must equal stop value at the boundary. Otherwise an arbitrage exists.

Economic meaning: 'indifference at the boundary.'

2. SMOOTH PASTING

$$\frac{\partial V}{\partial L}\Big|_{L^*(t)} = \frac{\partial}{\partial L}[NAV \cdot (1 - \pi)]\Big|_{L^*(t)}$$

Derivatives must also match — the value function pastes smoothly at the boundary.

Economic meaning: 'optimal' means no first-order gain from waiting longer.

Together: PDE + boundary conditions determine L(t) uniquely.*

Bellman's principle of optimality

Foundation of dynamic programming.

Principle

An optimal policy has the property that whatever the initial state and decision are, remaining decisions must be optimal from the new state

Equivalent

Optimal value function satisfies a recursive equation

Discrete time

$V(s,t) = \max_a [r(s,a) + \beta \cdot E[V(s',t+1) \mid s,a]]$

Continuous time analog

HJB equation (Hamilton-Jacobi-Bellman)

Why principle works

Optimal substructure: optimal solution contains optimal sub-solutions

Connection to GE-LAV

Optimal exit at time t depends only on current L_t, not history

Session 20 summary

What we accomplished today

1. Optimal exit timing = optimal stopping problem with state variable $L(t)$
2. Continuation value $V(L,t)$ satisfies the HJB equation in the holding region
3. At $L^*(t)$: value matching (V = stop value) + smooth pasting ($\partial V/\partial L = \partial \text{stop}/\partial L$)
4. Together: PDE + free-boundary conditions yield $L^*(t)$ uniquely. Numerical methods solve it.

Next session

Session 21: McKean-Vlasov equations and mean-field games — the math behind Failure 3

UNIT 5 · SESSION 20

From Bellman to HJB

Continuous-time derivation.

Setup

Maximize $\int f(s,a)\,dt + g(s_T)$ over policies a

Value function

$V(s,t) = \sup_a E[\int_t^T f + g(s_T) \mid s_t=s]$

Bellman recursion

$V(s,t) = \sup_a [f(s,a)\cdot dt + E[V(s+ds, t+dt)]]$

Apply Itô on E[V(...)]

$= V(s,t) + (\partial V/\partial t + \mu\cdot\partial V/\partial s + \frac{1}{2}\sigma^2\cdot\partial^2 V/\partial s^2)\cdot dt$

Cancel V(s,t) and divide by dt

$0 = \sup_a [f + \partial V/\partial t + \mu\cdot\partial V/\partial s + \frac{1}{2}\sigma^2\cdot\partial^2 V/\partial s^2]$

HJB equation

$\partial V/\partial t + \sup_a [f + L_a\cdot V] = 0$ where L_a is the infinitesimal generator

Optimal stopping: a special HJB

When the decision is when to act, not what action.

Setup

Continue or stop. Stop at any time τ chosen by the agent

Reward

If continuing: instantaneous $f(s)$; if stopping at τ: terminal $g(s_\tau)$

Value function

$V(s,t) = \max[g(s,t), \text{continuation value}]$

Continuation region

$\{(s,t) : V > g\}$

Stopping region

$\{(s,t) : V = g\}$

Free boundary

Optimal $s^*(t)$ separating continuation and stopping regions

Smooth-pasting (value-matching) principle

What pins down the free boundary.

Value matching

$V(s^*(t), t) = g(s^*(t), t)$ — values agree at the boundary

Smooth pasting

$\partial V/\partial s|_{(s^*(t))} = \partial g/\partial s|_{(s^*(t))}$ — derivatives agree

Why?

If derivatives differ, you can do better by adjusting τ slightly

Formal

Lemma 2.1, McKean (1965) or Karatzas-Shreve Ch. 2

In GE-LAV

These BCs pin down $L^*(t)$ given the HJB PDE

Practical

Numerical solver iterates until value matches and pastes smoothly

Worked example: American put option

The canonical optimal stopping problem.

Setup

American put: option to sell at strike K, exercise anytime

State

S_t (stock price), follows GBM

Exercise payoff

$\max(K - S_t, 0)$

Continuation value V

Solves HJB: $\partial V/\partial t + r\cdot S\cdot\partial V/\partial S + \frac{1}{2}\sigma^2 S^2\cdot\partial^2 V/\partial S^2 = rV$

Boundary

$S^*(t)$ such that $V(S^*(t),t) = K - S^*(t)$

In our context

$L^*(t)$ is the analog: 'exercise' = exit the fund

Numerical methods for HJB-stopping

How to actually solve.

Finite difference

Discretize (L,t) grid; solve PDE backward from terminal condition

Crank-Nicolson

Implicit; stable; second-order accurate

Penalty method

Add penalty for $V > g$; iterates until $V = \max(g, \text{continuation})$

Linear complementarity

Standard approach for free boundaries

Monte Carlo + LSMC

Longstaff-Schwartz for high-dimensional problems

Software

Python (FiPy, scipy) or commercial (FinDiff)

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Connection to GE-LAV: deriving L*(t)

Specific application to our setting.

State

L_t (the OU process)

Continuation reward

$CF(L_t, t)$ — operating cash flows · varies with L

Exercise payoff

$(1 - \pi(L_t, T-t))\cdot NAV$ — secondary market sale

HJB in continuation

$\partial V/\partial t + \kappa(\bar{L}-L)\cdot\partial V/\partial L + \tfrac{1}{2}\sigma^2\cdot\partial^2 V/\partial L^2 - r\cdot V + CF = 0$

Free boundary L*(t)

Found by smooth-pasting condition

In Session 25

Full derivation for split-track students

Stopping vs control problems

Different optimization structures.

Pure stopping

Only decide WHEN; one-shot decision

Pure control

Only decide HOW MUCH; allocation, hedging

Mixed

Both decisions simultaneously (e.g., LP also rebalances while holding)

HJB structure

Pure stopping → variational inequality; control → standard PDE

GE-LAV mixed

LP rebalances allocations (control) AND eventually exits (stopping)

Solvability

Mixed problems solvable in special cases; usually numerical

Bridge to Session 21

Individual decisions → collective dynamics.

HJB solves one agent's problem

Given the environment (π(L,T), L process)

Environment depends on others

π is itself determined by all LPs' decisions

Need: collective dynamics

How does everyone's decision interact?

Tool: mean-field games

Treat 'everyone else' as a distribution μ

Session 21

McKean-Vlasov and MFG · the mathematical bridge

Reading

Carmona-Delarue Ch. 1 · Lasry-Lions (2007)