UNIT 5 · MATHEMATICAL INTUITION BRIDGES

McKean-Vlasov & Mean-Field Games

Session 21 · The math behind Failure 3 — collective dynamics

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Primary Text: Liquidity Illusion (Forthcoming, 2026)

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What we'll cover today

1. **Why we need a mean-field framework**
   Recap: McKean-Vlasov externality

2. **McKean-Vlasov SDE**
   Distribution-dependent dynamics

3. **Mean-field game equilibrium**
   Best response → fixed point

4. **GE-LAV's MFG structure**
   How the framework instantiates this

5. **Solution methods preview**
   Fixed-point iteration

Mean-field game (MFG): the mathematical structure

An MFG is an equilibrium concept for problems where each agent's optimal decision depends on the distribution of all other agents' states.

Two coupled mathematical objects:

1. HJB EQUATION (PER AGENT)

Each LP solves their own optimal stopping problem (Session 20), but their value function depends on the cross-sectional distribution μ(L) of other LPs' states.

V(L, t; μ) ← each agent's value depends on μ

2. FOKKER-PLANCK EQ (DISTRIBUTION)

Given each LP's best-response decision rule, the cross-sectional distribution μ(L, t) evolves according to a Fokker-Planck equation.

μ(L,t) ← distribution depends on best-response policy

MFG equilibrium = fixed point: agents' best responses ⇄ resulting distribution

GE-LAV as a mean-field game

GE-LAV instantiates this MFG structure for private capital:

MFG component	GE-LAV instance	Reference
State variable	L(t) — liquidity state experienced by an individual LP	*OU process from Session 4*
Distribution	μ(L, t) — cross-sectional distribution of LP states	*Continuum of LPs assumed*
Control	Exit decision: continue or stop	*Optimal stopping (Session 20)*
Coupling	Secondary market price π(L, T-t) depends on μ	*More distressed sellers → higher discounts*
Equilibrium	Fixed point where each LP's best response is consistent with μ	*Existence proved Track 2 S30*

What 'collective dynamics' means in finance

Failure 3 of DCF: ignoring that LPs influence each other.

Individual decisions

Each LP picks an exit policy

Aggregate effect

Many LPs exiting → secondary supply rises → prices fall

Feedback

Falling prices → more LPs decide to exit (downward spiral)

Equilibrium concept

Self-consistent: each LP optimal given the resulting μ

DCF's failure

Treats each LP in isolation — no feedback term

MFG fixes this

Mathematical framework for self-consistent equilibrium

Session 21 summary

What we accomplished today

1. Mean-field game (MFG) is the natural framework for problems with cross-sectional coupling
2. Two coupled equations: HJB (per agent) + Fokker-Planck (distribution)
3. GE-LAV instantiates MFG: state = L(t), control = exit, coupling = secondary price $\pi(L,\mu)$
4. Existence of MFG equilibrium for GE-LAV proved in Track 2 Session 30

Next session

Session 22: Fokker-Planck equation in detail — how distributions evolve

McKean-Vlasov SDE: the building block

A new kind of stochastic process where the drift depends on the law.

Standard SDE

$dX = \mu(X,t)\cdot dt + \sigma(X,t)\cdot dW$

McKean-Vlasov SDE

$dX = \mu(X,t,\mu\_X)\cdot dt + \sigma(X,t,\mu\_X)\cdot dW$

Key difference

Drift/vol depend on the distribution μ_X of X itself

Example

$dL = \kappa(\bar{L}-L)\cdot dt + \sigma\cdot dW + \lambda\cdot[\text{avg}(L) - L]\cdot dt$

Interpretation

L drifts toward the cross-sectional mean of all LPs' L

Solvability

Existence/uniqueness under Lipschitz in (X, μ) · standard

Mean-field game (MFG) formulation

Game with infinitely many agents.

Setup

Continuum of identical agents · each picks policy α

Each agent's problem

max E[reward | own state X and population state μ]

Coupling

μ depends on all agents' policies α

Equilibrium

Self-consistent: each agent's best response, given μ_eq, generates μ_eq

Why 'mean-field'

Single agent treats others as average (mean) field

Lasry-Lions (2007)

Original theory · existence and uniqueness conditions

MFG vs N-player game: when do they coincide?

Convergence: large N → MFG.

Property	N-player game	MFG limit	Convergence rate
State dim	Each agent has own state	Single state X + distribution μ	—
Computational cost	Exponential in N	Polynomial in grid points	—
Equilibrium	Nash · may not exist	Lasry-Lions · exists generically	—
Empirical analog	Forex, exchange markets	PE secondary, OTC	—
Convergence rate	Symmetric N → ∞	MFG approximates Nash	O(1/√N)
For GE-LAV	10^4+ LPs → MFG accurate	—	Excellent

The MFG forward-backward system

Coupled PDEs for V and μ.

Backward HJB

For V: depends on μ (via the cost/reward structure)

Forward Fokker-Planck

For μ: depends on V (via the optimal policy α*)

Coupling

Solve jointly · fixed-point iteration

Master equation

Single equation in (V, μ) · Cardaliaguet-Cirant-Souganidis

Numerical

Iterate: solve HJB for V; solve FP for μ; check consistency

Convergence

Banach contraction · usually 10-20 iterations

Existence and uniqueness for MFG

Conditions under which equilibrium is well-posed.

Existence

Schauder fixed-point under continuity of (V,μ) coupling

Compactness

Bounded state space + bounded controls

Continuity

Coupling functionals continuous in Wasserstein metric

Uniqueness — Lasry-Lions condition

Monotonicity of coupling in μ

When fails

Coupling is non-monotone · multiple equilibria possible

In GE-LAV

Coupling π(L,T;μ) linear in moments of μ · monotone

Connection to GE-LAV specifically

How the MFG framework maps to private capital.

Agents

LPs (continuum, identical for now)

State X

L_t (LP's current liquidity state)

Policy α

Exit policy: when to sell on secondary

Distribution μ

Cross-sectional density of L across LPs

Coupling

Secondary price π(L,T;μ) depends on supply (function of μ)

Equilibrium

Self-consistent π and exit policy

What MFG explains that DCF can't

Empirical phenomena MFG predicts.

Crisis amplification

When many LPs distressed, secondary discount widens disproportionately

Fire sales

Mass simultaneous exits → π spike

Recovery patterns

After stress, π gradually returns to long-run · NOT linear

Cross-vintage spillover

2007 vintage stress affects 2005, 2009 prices too

Volume seasonality

Higher Q4 trading volume — pension fund recalibration

Empirically

All four predicted by GE-LAV and confirmed by Lazard data

Heterogeneous agents extension

When LPs differ — pension vs. endowment vs. SWF.

Standard MFG assumes identical agents

Single distribution μ

Heterogeneous version

Multiple types: each with own μ_type, optimal policy α_type

Coupling

Aggregate μ = weighted sum of type-specific μ_types

Complexity

Solvable in principle; numerically harder

Empirically

Pension fund 2008 behavior different from endowment 2008 behavior

Open research

Heterogeneous MFG · Carmona-Lacker (2018)

Bridge to Session 22

The Fokker-Planck equation governs μ. Let's study it.

MFG needs

FP equation for μ (forward density evolution)

Session 22 covers

FP equation derivation and properties

Why this matters

Half of MFG · the 'forward' part

Reading

Risken Ch. 4 or Pavliotis Ch. 4 (FP for OU)

After S22

Sessions 23-24 close Unit 5 · Jensen + Pigouvian

Then Unit 6

Split-track: cases + proofs combined