UNIT 5 · MATHEMATICAL INTUITION BRIDGES

The Fokker-Planck Equation

Session 22 · How distributions of LP states evolve through time

Samir Asaf, PhD, CFA, CMA, CTP, CM&AA

Senior Partner, Regent Financial LLC, New York

Former Finance Instructor, Stanford University

Primary Text: Liquidity Illusion (Forthcoming, 2026)

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What we'll cover today

1. **Forward Kolmogorov / Fokker-Planck**
   Distribution dynamics
2. **Derivation intuition**
   Conservation of probability
3. **Stationary solutions**
   Long-run distribution
4. **GE-LAV application**
   Cross-sectional LP state distribution
5. **Numerical methods preview**
   Finite-difference solvers

Fokker-Planck: how probability flows

For SDE $dL = \mu(L,t)dt + \sigma(L,t)dW$, the density $p(L,t)$ satisfies:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial L}[\mu(L,t)p] + \frac{1}{2}\frac{\partial^2}{\partial L^2}[\sigma^2(L,t)p]$$

Two terms with physical meaning:

Advection: $-\partial/\partial L\ [\mu(L,t)\cdot p]$	How the drift pulls probability mass along the L axis.
Diffusion: $\frac{1}{2}\cdot\partial^2/\partial L^2\ [\sigma^2(L,t)\cdot p]$	How random shocks spread out the probability mass.

What the FP equation describes

The forward partial differential equation for a density.

Given an SDE

$dX = \mu(X,t) \cdot dt + \sigma(X,t) \cdot dW$

Question

How does the density $p(X,t)$ of X_t evolve over time?

Answer

Fokker-Planck equation (also called forward Kolmogorov)

Discrete analog

Markov chain master equation

Why 'forward'

Evolves p forward in time from initial $p(x,0)$

Companion 'backward'

Backward equation for value functions (HJB)

Session 22 summary

What we accomplished today

1. Fokker-Planck (forward Kolmogorov) describes how SDE densities evolve through time
2. Two terms: advection (drift) + diffusion (noise spreading)
3. OU process has explicit stationary solution: Normal($\bar{L}$, $\sigma^2/2\kappa$)
4. GE-LAV uses Fokker-Planck for the cross-sectional LP distribution evolution

Next session

Session 23: Jensen's inequality and convexity — the math behind Failure 2

The FP equation: form

Standard form for an Itô diffusion.

PDE

$\partial p/\partial t = -\partial/\partial X[\mu(X,t)\cdot p] + \frac{1}{2}\cdot\partial^2/\partial X^2[\sigma^2(X,t)\cdot p]$

First term

Drift convection · $-\partial/\partial X[\mu\cdot p]$

Second term

Diffusion · $\frac{1}{2}\cdot\partial^2/\partial X^2[\sigma^2\cdot p]$

Conservation

$\int p\, dX = 1$ preserved (probability is conserved)

Initial condition

$p(X, 0) = p_0(X)$ · prior distribution

Boundary conditions

$p \to 0$ at infinity; absorbing or reflecting at finite boundaries

Derivation: via test functions

How to derive FP rigorously.

Step 1

Take an arbitrary smooth test function $f(X)$

Step 2

Compute $d/dt\, E[f(X_t)]$ using Itô

Step 3

Express as $\int f \cdot \partial p/\partial t\, dX$ (one side) = $\int (\mu \cdot f' + \frac{1}{2}\sigma^2 \cdot f'') \cdot p\, dX$ (other)

Step 4

Integrate by parts (twice on diffusion term)

Step 5

Use vanishing boundary terms (decay at infinity)

Step 6

Since f arbitrary, equate integrands → FP equation

FP for the OU process

Specialize to our case.

OU SDE

$dL = \kappa(\bar{L} - L)\cdot dt + \sigma\cdot dW$

FP equation

$\partial p/\partial t = \kappa\cdot\partial/\partial L[(L - \bar{L})\cdot p] + ½\sigma^2\cdot\partial^2 p/\partial L^2$

Stationary solution

$\partial p/\partial t = 0 \to p\_\infty(L) = \sqrt{(\kappa/\pi\sigma^2)} \cdot \exp(-\kappa(L-\bar{L})^2/\sigma^2)$

Recognize

$\text{Normal}(\bar{L}, \sigma^2/(2\kappa))$ — matches our prior derivation

Time-dependent

$p(L,t) \to$ Normal with mean $(L\_0-\bar{L})\cdot e\text{^}(-\kappa t) + \bar{L}$

Variance grows

$(\sigma^2/(2\kappa))\cdot(1 - e\text{^}(-2\kappa t))$

Stationary distribution: GE-LAV calibration

Numerical values for our parameters.

Parameter	Value	Interpretation
κ	0.45/yr	Mean reversion speed
σ	0.32	Diffusion volatility
$\bar{L}$	1.0	Long-run mean state
Stationary std	0.337	$= \sqrt{\sigma^2/2\kappa} \approx \sqrt{0.32^2/0.9}$
90% CI for L	[0.45, 1.55]	Most of distribution in normal range
Pr(L < -0.5)	0.07%	Tail probability
Pr(L < -1.5)	<0.001%	Effectively zero under stationary OU

Pr(crisis) under stationary OU is tiny — explains why we see crises rarely.

Spectral decomposition of FP operator

Eigenvalue analysis · convergence rates.

FP operator L_{FP}

$L_{FP} \cdot p = -\partial/\partial L[\mu \cdot p] + \frac{1}{2} \cdot \partial^2/\partial L^2[\sigma^2 \cdot p]$

Eigenvalue equation

$L_{FP} \cdot \varphi_n = -\lambda_n \cdot \varphi_n$

Smallest eigenvalue

$\lambda_0 = 0$ · stationary state

Second eigenvalue

$\lambda_1 = \kappa$ · the spectral gap

Implication

Half-life to stationarity = $\ln(2)/\kappa$

In our case

$\ln(2)/0.45 \approx 1.54$ yrs · the OU half-life

Numerical solution: finite difference

How to solve FP equation on a computer.

Discretize

L-grid: 1000 nodes from L_{min} to L_{max}

Time stepping

Crank-Nicolson (implicit · stable)

Sparse linear algebra

Tridiagonal system at each time step

Boundary handling

Reflecting BC at L_{min}, L_{max} for OU stability

Verification

Check $\int p \, dL = 1$ throughout

Long-time check

$p(L,t) \to N(\bar{L}, \sigma^2/(2\kappa))$ as $t \to \infty$

FP and HJB: the duality

Forward and backward equations are mathematical duals.

Backward HJB

$\partial V/\partial t + \sup_\alpha [b \cdot \partial V/\partial L + ½\sigma^2 \cdot \partial^2 V/\partial L^2 + r] = 0$

Forward FP

$\partial p/\partial t + \partial/\partial L[\alpha^* \cdot p] - ½\sigma^2 \cdot \partial^2 p/\partial L^2 = 0$

Duality

FP operator is adjoint of HJB's generator

Why duality matters

Solve HJB → know α^*; plug into FP → know μ; check consistency

Computational use

Forward-backward iteration · standard MFG algorithm

Master equation

Single equation combining both

Extensions: non-OU L processes

What if L doesn't follow OU?

Jump-diffusion L

FP gets non-local term · integro-PDE

Time-varying coefficients

κ(t), σ(t) · same FP structure, more numerics

CIR process

Cox-Ingersoll-Ross · always positive · FP still applies

Lévy-driven

Pure jump processes · FP becomes integral equation

Heston-like (two state)

Two coupled FP equations · 2D PDE

For GE-LAV

OU baseline + jump extension for crisis (open research)

Applications beyond GE-LAV

Where FP equations show up elsewhere.

Diffusion in physics

Heat equation = FP with no drift

Population dynamics

Density of biological populations

Plasma physics

Vlasov-FP equation

Asset pricing

Density of stochastic vol; underlying for Heston model

Optimal control

Backward equation for V, FP for state evolution

Cross-disciplinary

Same math, many applications

Bridge to Session 23

FP closes the math of how distributions evolve. Now: convexity.

S22 done

FP equation in our toolkit

Next math bridge

Jensen's inequality and convexity

Why convexity matters

Failure 2 of DCF: ignoring convex discount factors

Session 23 covers

Jensen inequality · why averages mislead

Practical impact

0.8-3.6% bias per year by asset class

Reading

Boyd-Vandenberghe Ch. 3 · convexity refresher