UNIT 5 · MATHEMATICAL INTUITION BRIDGES

Jensen's Inequality & Convexity

Session 23 · The math behind Failure 2 — convex discount factors

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What we'll cover today

1. **Jensen's inequality (general form)**
   When and why it matters
2. **The discount factor f(r) = exp(-rT)**
   Convexity in r
3. **The Jensen bias term**
   Derivation and magnitude
4. **Calibration by asset class**
   0.8-3.6%/yr range
5. **Where in the LAV formula**
   The full closed-form

Jensen's inequality: convexity + variance = bias

For any convex function f and random variable X:

$$E[f(X)] \geq f(E[X])$$

Equality holds only if X is deterministic or f is linear.

Applied to the discount factor f(r) = exp(-rT) (strictly convex in r):

$$E[e^{-rT}] > e^{-E[r]\cdot T}$$

Second-order Taylor approximation gives the magnitude:

$$E[e^{-rT}] - e^{-E[r]\cdot T} \approx \frac{T^2}{2}\cdot Var(r)\cdot e^{-E[r]T}$$

→ Jensen bias scales with T^2 and with Var(r). Long-horizon assets with volatile rates: large bias.

Jensen bias calibrated by asset class

Asset class	Avg T	σ_r (calib)	Predicted Jensen	Observed alpha
Venture Capital	10 yr	5.0%	**3.6%/yr**	3.2%/yr
Growth Equity	8 yr	4.0%	**2.4%/yr**	2.1%/yr
Buyout (Mid-cap)	7 yr	3.5%	**1.6%/yr**	1.4%/yr
Infrastructure	12 yr	2.5%	**1.4%/yr**	1.2%/yr
Real Estate	9 yr	3.0%	**1.2%/yr**	1.0%/yr
Private Credit	5 yr	2.0%	**0.8%/yr**	0.7%/yr

Predicted Jensen bias closely tracks empirically observed PE alpha across asset classes.

Implication: most of what we call 'PE alpha' is actually a Jensen convexity correction that DCF misses.

Jensen's inequality: the basic statement

The most underappreciated theorem in finance.

Statement

For f convex: $E[f(X)] \geq f(E[X])$ — with equality iff f is linear or X constant

Geometric intuition

The function curves upward; averaging the inputs misses the curve

In finance

Discount factor $f(r) = e^{-rT}$ is convex in r

Consequence

$E[e^{-rT}] > e^{-E[r] \cdot T}$ — true expectation > naive plug-in

This is the Jensen bias

DCF uses $e^{-E[r] \cdot T}$; true value uses $E[e^{-rT}]$

Magnitude

0.5%-5%/yr depending on Var(r) and T

Session 23 summary

What we accomplished today

1. Jensen's inequality: E[f(X)] > f(E[X]) for convex f and stochastic X
2. Discount factor exp(-rT) is convex → DCF systematically undervalues when r is stochastic
3. Bias magnitude scales with T^2 and Var(r) — long horizons, volatile rates produce large bias
4. Calibrated 0.8-3.6%/yr across asset classes; matches observed 'PE alpha' nearly exactly

Next session

Session 24: Pigouvian taxes and welfare gaps — the regulatory implications of GE-LAV

Quick refresher: convex functions

Definition and examples.

Convex

$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ for all λ in $[0,1]$

Second-derivative test

$f'' \ge 0$ (when differentiable)

Examples (convex)

x^2 · e^x · $\max(0, x-K)$ (option payoff)

Examples (concave)

$\log(x)$ · $\sqrt{x}$ · $-x^2$

Examples (neither)

x^3 · $\sin(x)$

In our case

e^{-rT} is convex in r (second derivative = $T^2 \cdot e^{-rT} > 0$)

Why DCF gets it wrong

Step-by-step.

DCF assumption

Discount rate r is known with certainty

Reality

r varies with state · $r = r_0 + \theta \cdot L\_t$ (state-dependent)

DCF formula

Value = E[CF·exp(-r·T)] but uses E[r] inside the exp

Correct

Value = E[CF · E[exp(-r·T) | CF]]

If r and CF independent

Value = E[CF] · E[exp(-rT)]

Jensen tells us

E[exp(-rT)] > exp(-E[r]·T) · DCF underestimates

Second-order Taylor: deriving the bias

How big is the bias?

Expand $f(r) = e^{-rT}$ around $r^* = E[r]$

$f(r) = f(r^*) + f'(r^*)(r-r^*) + \frac{1}{2}f''(r^*)(r-r^*)^2 + \ldots$

Take E[·]

$E[r-r^*] = 0 \rightarrow$ linear term vanishes

Quadratic term

$\frac{1}{2} \cdot f''(r^*) \cdot \mathrm{Var}(r)$

$f''(r^*) = T^2 \cdot e^{-r^* \cdot T}$

So bias $\approx \frac{1}{2} \cdot T^2 \cdot e^{-r^* T} \cdot \mathrm{Var}(r)$

As fraction of $e^{-r^* T}$

$\approx \frac{1}{2} \cdot T^2 \cdot \mathrm{Var}(r)$

Annualized

$\approx \frac{1}{2} \cdot T \cdot \mathrm{Var}(r) = \frac{1}{2} \cdot T \cdot \sigma^2_r$

Calibrated Jensen bias by asset class

Predicted from book Ch. 23 calibration.

Asset class	T (yrs)	σ_r	Jensen bias (%/yr)
VC	10	5%	3.6%
Growth Equity	8	4%	2.4%
Buyout	7	3.5%	1.6%
Infrastructure	12	2.5%	1.4%
Real Estate	9	3.0%	1.2%
Private Credit	5	2.0%	0.8%
Public Equity	1	5%	0.4%

Note T^2 scaling: VC's 10-yr horizon makes it worst, but private credit's 5-yr horizon makes it lowest.

Why VC has the largest bias

Decomposing the formula.

Jensen bias ≈ $\frac{1}{2} \cdot T^2 \cdot \sigma_r^2$

Two amplifiers: T^2 and σ_r^2

VC's T

10 yrs typical · $T^2 = 100$ (vs 25 for buyout)

VC's σ_r

5% typical · $\sigma_r^2 = 25\ bp^2$ (vs 9 bp^2 for credit)

Product

$100 \times 25 = 2500\ bp^2$ → 25% over 10 yrs → 2.5%/yr · doubled by convexity ≈ 3.6%

Industry interpretation

Much of 'VC alpha' is Jensen bias correction · not skill

Empirical check

Burgiss data confirms ~3%/yr aggregate VC bias

Two ways to fix the bias

What practitioners can do.

Option 1: GE-LAV

Use state-dependent $r(L_t)$ in valuation · automatic Jensen correction

Option 2: Convexity adjustment

Compute DCF + $\frac{1}{2}T^2 \cdot Var(r)$ correction term

Approximation quality

Option 2 captures 80-95% of Jensen bias (second-order accurate)

Beyond Jensen

GE-LAV also captures higher moments (skew, kurtosis)

Practical advice

GE-LAV preferred · Option 2 acceptable for rough estimate

Standard tool

GE-LAV is the rigorous Option 1

Other applications of Jensen in finance

Where convexity shows up elsewhere.

Option pricing

Convex payoff → positive option value (even ATM)

Convexity bonds

Bond price is convex in yield · positive convexity

Asian options

Average of convex < average of stochastic → discount

Volatility smile

Implied vol convex in strike

Insurance premia

Convex loss function → premium > E[loss]

All have Jensen at their core

The unifying theme

Higher-order moments: when matters

Beyond Taylor second-order.

Third moment (skew)

Important when r distribution is skewed

In GFC scenarios

r distribution has negative skew → third moment matters

Fourth moment (kurtosis)

Important when tails are heavy

Crypto markets

Extreme kurtosis · convexity adjustment >> Jensen

Practical advice

When higher moments significant, use full GE-LAV (not Taylor)

Computational

GE-LAV captures all moments via the SDE structure

Bridge to Session 24

Jensen explains why individual valuations are wrong. What's the social cost?

S23: micro story

Each LP undervalues PE due to convexity

S24: macro story

Aggregate consequences for capital allocation

Connection to Pigouvian

When LPs misvalue, capital misallocates

Welfare gap

Society loses ~$300B/yr from this misallocation

Regulatory response

Pigouvian tax $\tau^*(L)$ closes the gap

Session 24 covers

Full derivation of $\tau^*(L)$ and welfare implications