UNIT 6 · SPLIT TRACK: T1 PRACTITIONER / T2 RESEARCHER

Infrastructure Case / Fokker-Planck & Master Eq

Session 27 · Macquarie infra case · density evolution

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Primary Text: Liquidity Illusion (Forthcoming, 2026)

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What we'll cover today

1. **Recap and track choice**
   S26 deliverables · today's split
2. **Track 1: Infrastructure asset case**
   20-year toll road project
3. **Track 1: Cash flow timing & state**
   Why infra Jensen bias matters
4. **Track 1 deliverable**
   Asset-level GE-LAV valuation memo
5. **Track 2: Fokker-Planck derivation**
   Density evolution for OU
6. **Track 2: Master equation**
   Coupled MFG density-value system

Recap: Session 26 · VC & Secondaries / MFG Existence

Three takeaways carried forward:

1. VC has the largest Jensen bias ($T=10$ yrs, $\sigma\_r=5\%$ → 5%/yr correction)

2. MFG existence proven via Schauder fixed-point on (μ,π,V) triple

3. Uniqueness holds under Lasry-Lions monotonicity (verified for our calibration)

Today: infrastructure (T1) · density dynamics (T2)

Today the class divides. Choose your seat:

TRACK 1 — Practitioner

20-yr horizon means T^2 scaling matters most. Apply GE-LAV term-structure $\pi(L,T)$ to a toll road project. Quantify the DCF undervaluation.

TRACK 2 — Researcher

Derive the Fokker-Planck (forward Kolmogorov) equation. Connect to the GE-LAV master equation coupling V and μ.

Reminder: Infrastructure is where Jensen bias is largest in absolute dollars — high-stakes for pension fund LPs.

Track 1 · Case: Macquarie toll-road infrastructure fund

Asset: 35-mile toll road, US Northeast corridor. Acquired 2010, 25-yr concession.

Parameter	Value	Source
Acquisition price	\$1.2B equity + \$1.8B debt	Macquarie 10-K
Initial cash yield	5.2% (Year 1)	Operating proforma
Concession maturity	2035 (25 years)	State DOT contract
Annual revenue growth	CPI + 2% (escalator)	Concession terms
Operating margin	62% (Year 1) → 73% (Year 10)	Scale economies
Cost of equity (DCF)	9.5% real	WACC build-up
Constant π (DCF)	2.0%	Long-duration premium

Track 1 · DCF vs GE-LAV with 20-year horizon

How the term structure π(L,T) changes valuation at long horizons.

Year	DCF NPV	GE-LAV NPV (L=0)	GE-LAV NPV (L=-0.5)	GE-LAV NPV (L=-1.5)
Year 5	$1.41B	$1.32B	$1.05B	$0.62B
Year 10	$1.86B	$1.69B	$1.32B	$0.71B
Year 15	$2.31B	$2.04B	$1.55B	$0.78B
Year 20	$2.76B	$2.34B	$1.71B	$0.81B
Year 25 (exit)	$3.20B	$2.61B	$1.85B	$0.84B
Total NPV (DCF)	$2.94B	$2.46B	$1.86B	$0.97B
DCF gap	0%	-16%	-37%	-67%

Long-horizon infra is where the term-structure correction has the largest absolute dollar impact.

Track 1 · Deliverable: asset-level GE-LAV valuation memo

Your task: produce a single-asset valuation memo for the toll road, accounting for time-varying L(t).

Length

3 pages, asset valuation memo format

Method

Use $\pi(L,T)$ term structure from book Chapter 7

Sensitivity

Show NPV under L=0, L=-0.5, L=-1.5

Hedging

Identify hedges (10-yr Treasury, REIT index) for long-duration L exposure

Deliverable due

End of Session 28 · upload via course site

Track 2 · Fokker-Planck (forward Kolmogorov) equation

For $dL = \mu(L,t)dt + \sigma(L,t)dW$, the density $p(L,t)$ evolves by:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial L}[\mu(L,t)\,p] + \frac{1}{2}\frac{\partial^2}{\partial L^2}[\sigma^2(L,t)\,p]$$

Specialized to OU: $\mu = \kappa(\bar{L}-L)$, σ constant ⇒

$$\frac{\partial p}{\partial t} = \kappa\frac{\partial}{\partial L}[(L-\bar{L})p] + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial L^2}$$

Stationary solution: $\partial p/\partial t = 0$ ⇒ Normal($\bar{L}$, $\sigma^2/2\kappa$).

Verify directly by substituting Gaussian density.

Reference: Risken, The Fokker-Planck Equation, Ch. 4-5.

Track 2 · Derivation: Itô + integration by parts

Outline: for any smooth test function f, compute d/dt E[f(L_t)]

Step 1	$E[f(L_t)] = \int f(L)\, p(L,t)\, dL$
Step 2	By Itô: $df = f' \cdot dL + \frac{1}{2} f'' \cdot (dL)^2 \rightarrow E[df]/dt = E[\mu \cdot f' + \frac{1}{2}\sigma^2 \cdot f'']$
Step 3	Differentiate the integral and equate: $\int f \cdot \partial p/\partial t\, dL = \int (\mu \cdot f' + \frac{1}{2}\sigma^2 \cdot f'') \cdot p\, dL$
Step 4	Integrate by parts (twice for the second term), using vanishing boundary terms
Step 5	Since f arbitrary, equate integrands: get Fokker-Planck

Track 2 · The GE-LAV master equation

Combine HJB (for V) and Fokker-Planck (for μ): a coupled system.

HJB (backward):

$$\partial_t V + \kappa(\bar{L} - L)\partial_L V + \frac{\sigma^2}{2}\partial_{LL} V - rV + CF(L, t; \mu) = 0$$

Fokker-Planck (forward):

$$\partial_t \mu = -\partial_L[\alpha^*(L, t; V)\mu] + \frac{\sigma^2}{2}\partial_{LL}\mu$$

The coupling: V depends on μ (via CF). μ depends on V (via α*).

Self-consistent solution = MFG equilibrium. Existence proven Session 26.

References: Lasry-Lions (2007) JFA; Carmona-Delarue (2018) Vol I, Ch. 4.

Track 2 · Problem set (due Session 28)

P1

Verify the OU stationary density is Normal($\bar{L}$, $\sigma^2/2\kappa$)

P2

Compute the half-life to stationarity (use spectral gap of FP operator)

P3

Show: if α^* is bounded and Lipschitz, FP equation has a unique density solution

P4

Derive the master equation for the case $\pi(L,T\text{-}t;\mu) = a + b\cdot L + c\cdot\int L' d\mu(L')$

P5

(Bonus) Implement a finite-volume FP solver in Python

Track 1 · Hedging instruments for long-duration infra

Once GE-LAV reveals exposure, what can the LP actually do?

Long-term rates

10-30yr Treasury futures or swaps to hedge duration

Inflation

TIPS or inflation swaps (concession revenues are CPI-linked)

Cyclical L exposure

REIT index short — real estate L is correlated with infra L

Default risk on toll volume

CDS on the asset's debt tranche (where available)

Cost of hedge

Typically 0.3-0.8% per annum drag · still positive NPV vs unhedged

Best practice

Hedge ~30-50% of exposure; full hedge often dominated by partial

Both tracks reconvene: what we agree on

After today, Tracks 1 and 2 agree on:

Track 1 produced

An infrastructure valuation that accounts for long-horizon term-structure — 16-67% adjustment over DCF.

Track 2 produced

The Fokker-Planck equation governing μ, and the coupled master equation that defines the GE-LAV equilibrium.

Common ground

Long-duration assets feel the L(t) state most acutely. T1 quantifies the dollars; T2 explains the math of how the distribution of L states evolves.

Track 2 · Boundary conditions for the FP equation

FP equation alone is under-determined. Need boundary and initial conditions.

Initial condition

$p(L, 0) = \delta(L - L_0)$ — current LP state with certainty

Far-field decay

$p(L, t) \to 0$ as $|L| \to \infty$ (probability vanishes far from $\bar{L}$)

Conservation

$\int p(L, t)\, dL = 1$ for all t (total probability conserved)

Reflecting boundary

Used if L bounded by $\pm L_{max}$ (some calibrations)

Absorbing boundary

Used when LP exits permanently (one-shot stopping)

Practical solver

Crank-Nicolson on a finite L-grid · 1000 nodes typical

Session 27 summary

What we accomplished today

1. Infrastructure has the largest absolute-dollar Jensen + term-structure correction
2. Track 1 computed the GE-LAV adjustment for a 25-yr toll road: 16% (normal) to 67% (crisis)
3. Track 2 derived the Fokker-Planck equation and connected it to the GE-LAV master equation
4. The master equation couples V (HJB) and μ (FP) — fixed-point solution is the MFG equilibrium

Next session

Session 28: Private Credit / GE Equilibrium Existence